Journal of Applied Mechanics and Technical Physics, Vol. 46, No. 4, pp. 564–569, 2005

GENERALIZATION OF VLASOV'S EQUATIONS FOR A CYLINDRICAL SHELL TO THE CASE OF A TRANSVERSELY ISOTROPIC MATERIAL

A. B. Nerubailo and B. V. Nerubailo

UDC 539.3

Differential equations of the general theory of transversely isotropic cylindrical shells are obtained; in a certain sense, these equations are generalizations of Vlasov's and Ambartsumyan's equations. This allowed us on the basis of Novozhilov's criterion (comparison of variability of the stress state in the principal orthogonal directions) to divide the initial equations according to Goldenweiser into approximate equations of the type of the semi-momentless theory, theory of the edge effect and flexural state, which are also generalizations of equations that describe the elementary stress states of an isotropic shell. Numerical values are found for criteria of matching of approximate equations that describe the elementary stress states in the asymptotic synthesis of the full stress state. Examples of calculations and experimental data for a shell with and without allowance for transverse shear strain are given.

Key words: *cylindrical shells, stress state, Vlasov's equation.*

The stress–strain state of a circular cylindrical shell under the action of a normal load arbitrarily distributed over the shell surface is considered. In a particular case, this can be a locally applied or a concentrated force.

1. The shell is assumed to be made of a transversely isotropic material in which the plane of isotropy at each point is parallel to the mid-surface, and the principal direction of elasticity perpendicular to the plane of isotropy coincides at each point of the shell with the corresponding normal γ.

The following assumptions are made:

— the shear stresses on the areas perpendicular to the mid-surface or the corresponding shear strains over the shell thickness vary in accordance with a prescribed law;

— the displacement normal to the mid-surface is independent of the coordinate over the shell thickness;

— the normal stresses in the areas parallel to the mid-surface are neglected.

Mathematically, this means that

$$\tau_{\alpha\gamma} = (h^2/8 - \gamma^2/2)\varphi(\alpha,\beta), \qquad \tau_{\beta\gamma} = (h^2/8 - \gamma^2/2)\psi(\alpha,\beta),$$
$$\varepsilon_\gamma = 0, \qquad w = w(\alpha,\beta), \qquad \sigma_\gamma \approx 0, \tag{1.1}$$

where $w(\alpha,\beta)$, $\varphi(\alpha,\beta)$, and $\psi(\alpha,\beta)$ are the sought functions of the dimensionless longitudinal (α) and circumferential (β) coordinates.

Differential equations are obtained, which are generalizations of Vlasov's [1] and Ambartsumyan's [2] equations in a certain sense and which can be called the differential equations of the general theory of transversely isotropic shells. This allowed us on the basis of Novozhilov's criterion (comparison of variability of the stress state in the principal orthogonal directions) [3] to divide the initial equations according to Goldenweiser [4] into approximate equations, which are also generalizations of equations that describe the elementary stress states. The previously formulated methods of asymptotic synthesis (MAS) of the stress–strain state [5, 6] are generalized here to the case of shells made of a transversely isotropic material.

Moscow State Aviation Institute (Technical University), Moscow 125871; prof_nebo@mail.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 46, No. 4, pp. 125–132, July–August, 2005. Original article submitted March 22, 2004; revision submitted June 28, 2004.

In contrast to the problem formulation used in [2], we do not impose constraints on the shell length, its depth, and variability of the external load and the stress–strain state. Then, for internal forces, flexural moment, and torque, with allowance for the accepted hypotheses (1.1), we obtain the following differential dependences generalizing their analogs in [1, 2]:

$$T_1 = \frac{Eh}{R}\Big[\frac{\partial u}{\partial \alpha} + \nu\Big(\frac{\partial v}{\partial \beta} - w\Big) + c^2\,\frac{\partial^2 w}{\partial \alpha^2}\Big], \qquad T_2 = \frac{Eh}{R}\Big[\frac{\partial v}{\partial \beta} - w + \nu\,\frac{\partial u}{\partial \alpha} - c^2\Big(\frac{\partial^2 w}{\partial \beta^2} + w\Big)\Big],$$

$$S_1 = \frac{Eh}{2(1+\nu)R}\Big(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\,c^2\,\frac{\partial^2 w}{\partial \alpha\,\partial \beta}\Big), \qquad S_2 = \frac{Eh}{2(1+\nu)R}\Big(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} - c^2\,\frac{\partial^2 w}{\partial \alpha\,\partial \beta}\Big),$$

$$G_1 = -\frac{D}{R^2}\Big[\Big(\frac{\partial^2}{\partial \alpha^2} + \nu\,\frac{\partial^2}{\partial \beta^2}\Big)w + \frac{\partial u}{\partial \alpha} + \nu\,\frac{\partial v}{\partial \beta} - \frac{h^2 R}{10G^\gamma}\Big(\frac{\partial \varphi}{\partial \alpha} + \nu\,\frac{\partial \psi}{\partial \beta}\Big)\Big],$$

$$G_2 = -\frac{D}{R}\Big[\Big(\frac{\partial^2}{\partial \beta^2} + \nu\,\frac{\partial^2}{\partial \alpha^2}\Big)w + w - \frac{h^2 R}{10G^\gamma}\Big(\frac{\partial \psi}{\partial \beta} + \nu\,\frac{\partial \varphi}{\partial \alpha}\Big)\Big], \tag{1.2}$$

$$G_{12} = (1-\nu)\frac{D}{R^2}\Big[\frac{\partial v}{\partial \alpha} + \frac{\partial^2 w}{\partial \alpha\,\partial \beta} - \frac{1}{2}\,\frac{h^2}{10G^\gamma}\Big(\frac{\partial \psi}{\partial \alpha} + \frac{\partial \varphi}{\partial \beta}\Big)\Big],$$

$$G_{21} = (1-\nu)\frac{D}{R^2}\Big[\frac{1}{2}\Big(\frac{\partial v}{\partial \alpha} - \frac{\partial u}{\partial \beta}\Big) + \frac{\partial^2 w}{\partial \alpha\,\partial \beta} - \frac{1}{2}\,\frac{h^2}{10G^\gamma}\Big(\frac{\partial \psi}{\partial \alpha} + \frac{\partial \varphi}{\partial \beta}\Big)\Big],$$

$$Q_1 = h^3\varphi/12, \qquad Q_2 = h^3\psi/12.$$

In these relations, which are, as was already noted, generalizations of the equations of the general theory of isotropic shells [1] and equations of gently sloping, transversely isotropic shells [2], we use the following notation: E is Young's modulus for directions in the plane of isotropy, ν is Poisson's ratio characterizing the shrinkage in the plane of isotropy under extension in the same plane, and G^γ is the shear modulus for planes normal to the plane of isotropy.

Note that the mechanical characteristics of the shell material in orthogonal planes are related by the known expression $\nu^{\gamma\gamma}E^\gamma = \nu^\gamma E$, where E^γ is Young's modulus for directions perpendicular to the plane of isotropy, ν^γ is Poisson's ratio characterizing the shrinkage in the plane of isotropy under extension in the direction perpendicular to this plane, and $\nu^{\gamma\gamma}$ is Poisson's ratio characterizing the shrinkage in the direction perpendicular to the plane of isotropy under extension in the plane of isotropy.

Substituting Eqs. (1.2) into the equilibrium equations, we obtain the following system of differential equations for five unknown functions $u(\alpha,\beta)$, $V(\alpha,\beta)$, $w(\alpha,\beta)$, $\varphi(\alpha,\beta)$, and $\psi(\alpha,\beta)$:

$$\Big(\frac{\partial^2}{\partial \alpha^2} + \frac{1-\nu}{2}\,\frac{\partial^2}{\partial \beta^2}\Big)u + \frac{1+\nu}{2}\,\frac{\partial^2 v}{\partial \alpha\,\partial \beta} - \Big[\nu\,\frac{\partial}{\partial \alpha} - c^2\Big(\frac{\partial^3}{\partial \alpha^3} - \frac{1-\nu}{2}\,\frac{\partial^3}{\partial \alpha\,\partial \beta^2}\Big)\Big]w = 0,$$

$$\frac{1+\nu}{2}\,\frac{\partial^2 u}{\partial \alpha\,\partial \beta} + \Big(\frac{\partial^2}{\partial \beta^2} + \frac{1-\nu}{2}\,\frac{\partial^2}{\partial \alpha^2}\Big)v - \Big(\frac{\partial}{\partial \beta} - \frac{3-\nu}{2}\,c^2\,\frac{\partial^3}{\partial \alpha^2 \partial \beta}\Big)w - \frac{h^3}{12}\psi = 0,$$

$$\nu\,\frac{\partial u}{\partial \alpha} + \frac{\partial v}{\partial \beta} - w - c^2\Big(\frac{\partial^2 w}{\partial \beta^2} + w\Big) + \frac{1-\nu^2}{12}\,\frac{h^2 R}{E}\Big(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta}\Big) = -\frac{(1-\nu^2)R^2}{Eh}\,p(\alpha,\beta),$$

$$(1-\nu)\,\frac{\partial^2 v}{\partial \alpha^2} + \frac{\partial}{\partial \beta}\,(\nabla^2 w + w) - \frac{h^2 R}{10G^\gamma}\Big(\frac{\partial^2 \psi}{\partial \beta^2} + \frac{1-\nu}{2}\,\frac{\partial^2 \psi}{\partial \alpha^2} + \frac{1+\nu}{2}\,\frac{\partial^2 \varphi}{\partial \alpha\,\partial \beta}\Big) + (1-\nu^2)\,\frac{R^3}{E}\psi = 0, \tag{1.3}$$

$$\frac{\partial^2 u}{\partial \alpha^2} - \frac{1-\nu}{2}\,\frac{\partial^2 u}{\partial \beta^2} + \frac{1+\nu}{2}\,\frac{\partial^2 v}{\partial \alpha\,\partial \beta} + \frac{\partial}{\partial \alpha}\,\nabla^2 w$$

$$-\frac{h^2 R}{10G^\gamma}\Big(\frac{\partial^2 \varphi}{\partial \alpha^2} + \frac{1-\nu}{2}\,\frac{\partial^2 \varphi}{\partial \beta^2} + \frac{1+\nu}{2}\,\frac{\partial^2 \psi}{\partial \alpha\,\partial \beta}\Big) + (1-\nu^2)\,\frac{R^3}{E}\,\varphi = 0.$$

System (1.3) consisting of five resolving equations of the tenth order can be converted to a form accepted in the general theory of isotropic shells [1]:

$$\frac{\partial^2 u}{\partial\alpha^2}+\frac{1-\nu}{2}\,\frac{\partial^2 u}{\partial\beta^2}+\frac{1+\nu}{2}\,\frac{\partial^2 v}{\partial\alpha\,\partial\beta}-\nu\,\frac{\partial w}{\partial\alpha}+c^2\Big(\frac{\partial^3 w}{\partial\alpha^3}-\frac{1-\nu}{2}\,\frac{\partial^3 w}{\partial\alpha^2\,\partial\beta}\Big)=0,$$

$$\frac{1+\nu}{2}\,\frac{\partial^2 u}{\partial\alpha\,\partial\beta}+\frac{\partial^2 v}{\partial\beta^2}+\frac{1-\nu}{2}\,\frac{\partial^2 v}{\partial\alpha^2}-\frac{\partial w}{\partial\beta}+\frac{3-\nu}{2}\,c^2\,\frac{\partial^3 w}{\partial\alpha\,\partial\beta}=\frac{h^3}{12}\,\psi, \tag{1.4}$$

$$\nabla^2\nabla^2 w+\frac{\partial^2 w}{\partial\beta^2}+\frac{\partial^3 u}{\partial\alpha^3}-\frac{1-\nu}{2}\,\frac{\partial^3 u}{\partial\alpha\,\partial\beta^2}+\frac{3-\nu}{2}\,\frac{\partial^3 v}{\partial\alpha^2\,\partial\beta}$$

$$-\frac{12R^2}{h^2}\,(1-\Omega\nabla^2)\Big[\nu\,\frac{\partial u}{\partial\alpha}+\frac{\partial v}{\partial\beta}-w-c^2\Big(\frac{\partial^2 w}{\partial\beta^2}+w\Big)\Big]=\frac{R^4}{D}\,(1-\Omega\nabla^2)p(\alpha,\beta).$$

In these equations, the parameter $\Omega=Eh^2/(10(1-\nu^2)G^\gamma R^2)$ characterizes the influence of transverse shear on the stress–strain state of the shell and D is the cylindrical rigidity.

System (1.4) should be supplemented by a system of two differential equations relating the sought functions φ and ψ with the displacements u, v, and w:

$$\Big[(1-\nu^2)\,\frac{R^3}{E}-\frac{h^2R}{10G^\gamma}\,\nabla^2\Big]\Big(\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}\Big)+\nabla^2\nabla^2 w+\frac{\partial^2 w}{\partial\beta^2}+\frac{\partial^3 u}{\partial\alpha^3}-\frac{1-\nu}{2}\,\frac{\partial^3 u}{\partial\alpha\,\partial\beta^2}+\frac{3-\nu}{2}\,\frac{\partial^3 v}{\partial\alpha^2\,\partial\beta}=0, \tag{1.5}$$

$$\nu\,\frac{\partial u}{\partial\alpha}+\frac{\partial v}{\partial\beta}-w-c^2\Big(\frac{\partial^2 w}{\partial\beta^2}+w\Big)+\frac{1-\nu^2}{12}\,\frac{Rh^2}{E}\Big(\frac{\partial\varphi}{\partial\alpha}+\frac{\partial\psi}{\partial\beta}\Big)=-\frac{(1-\nu^2)R^2}{Eh}\,p(\alpha,\beta).$$

The first equation in (1.5) is obtained on the basis of the fourth and fifth equations in (1.3); the second equation in (1.5) coincides with the third equation of system (1.3).

To obtain approximate equations corresponding to a prescribed variability of the stress–strain state, it is of undoubted interest to reduce system (1.4) to a classical form suggested by Vlasov [7]:

$$\nabla^2\nabla^2 u=-\nu\,\frac{\partial^3 w}{\partial\alpha^3}+\frac{\partial^3 w}{\partial\alpha\,\partial\beta^2}+c^2\Big(\frac{\partial^5 w}{\partial\alpha^5}-\frac{\partial^5 w}{\partial\alpha\,\partial\beta^4}\Big),$$

$$\nabla^2\nabla^2 v=-(2+\nu)\frac{\partial^3 w}{\partial\alpha^2\,\partial\beta}-\frac{\partial^3 w}{\partial\beta^3}+2c^2\Big(\frac{\partial^5 w}{\partial\alpha^4\,\partial\beta}+\frac{\partial^5 w}{\partial\alpha^2\,\partial\beta^3}\Big), \tag{1.6}$$

$$\nabla^2\nabla^2(\nabla^2+1)^2 w-2(1-\nu)\Big(\frac{\partial^4 w}{\partial\alpha^4}-\frac{\partial^4 w}{\partial\alpha^2\,\partial\beta^2}\Big)\nabla^2 w+\frac{1-\nu^2}{c^2}\,(1-\Omega\nabla^2)\frac{\partial^4 w}{\partial\alpha^4}=\frac{R^4}{D}\,(1-\Omega\nabla^2)\nabla^2\nabla^2 p(\alpha,\beta).$$

We can easily see that the first two equations of system (1.6) completely coincide with Vlasov's equations [7], whereas the third equation contains additional terms that take into account transverse shear strain in the shell.

2. The resultant system (1.6) is applicable for solving problems of transversely isotropic cylindrical shells with a load arbitrarily distributed over the shell surface (without constraints on the shell size and load variability).

We consider equations simplified in terms of the criterion of load variability, which can yield applicable results in analyzing particular problems.

In the case of a stress–strain state corresponding to the criterion $|\partial^2 f/\partial\beta^2|\gg|\partial^2 f/\partial\alpha^2|$ (f is an arbitrary unknown function: displacement, force, moment, etc.), on the basis of (1.6), we obtain the following equation for the radial displacement $w(\alpha,\beta)$:

$$\Big(1-\Omega\,\frac{\partial^2}{\partial\beta^2}\Big)\frac{\partial^4 w}{\partial\alpha^4}+\frac{c^2}{1-\nu^2}\,\frac{\partial^4}{\partial\beta^4}\Big(\frac{\partial^2}{\partial\beta^2}+1\Big)^2 w=\frac{R^2}{Eh}\Big(1-\Omega\,\frac{\partial^2}{\partial\beta^2}\Big)\frac{\partial^4}{\partial\beta^4}\,p(\alpha,\beta). \tag{2.1}$$

It follows from (2.1) that the radial displacement in the mechanical-mathematical model of a transversely isotropic shell proposed here is a resolving function in terms of which all force and strain factors of the shell are expressed with the help of differential operators:

$$\frac{\partial^2 u}{\partial\beta^2}=\frac{\partial w}{\partial\alpha},\qquad \frac{\partial v}{\partial\beta}=w; \tag{2.2}$$

$$T_1=\frac{Eh}{(1-\nu^2)R}\Big[\frac{\partial u}{\partial\alpha}+\nu\Big(\frac{\partial v}{\partial\beta}-w\Big)\Big],\qquad T_2=\frac{Eh}{(1-\nu^2)}\Big(\frac{\partial v}{\partial\beta}-w+\nu\,\frac{\partial u}{\partial\alpha}\Big),$$

$$S_1 = S_2 = S = \frac{Eh}{2(1+\nu)R}\Big(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\Big),$$

$$G_1 = -\frac{D}{R^2}\Big[\nu\Big(\frac{\partial^2 w}{\partial \beta^2} + w\Big) - \frac{h^2R^2}{10G^\gamma}\Big(\frac{\partial \varphi}{\partial \alpha} + \nu\,\frac{\partial \psi}{\partial \beta}\Big)\Big], \quad G_2 = -\frac{D}{R^2}\Big[\frac{\partial^2 w}{\partial \beta^2} + w - \frac{h^2R^2}{10G^\gamma}\Big(\frac{\partial \psi}{\partial \beta} + \nu\,\frac{\partial \varphi}{\partial \alpha}\Big)\Big],$$

$$Q_1 = h^3\varphi/12, \qquad Q_2 = h^3\psi/12. \tag{2.3}$$

Instead of (1.5), the differential equations relating φ and ψ with displacements now acquire the form

$$\Big[(1-\nu^2)\,\frac{R^3}{E} - \frac{h^2R}{10G^\gamma}\,\frac{\partial^2}{\partial \beta^2}\Big]\Big(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta}\Big) + \frac{\partial^4 w}{\partial \beta^4} + \frac{\partial^2 w}{\partial \beta^2} = 0,$$

$$\nu\,\frac{\partial u}{\partial \alpha} + \frac{\partial v}{\partial \beta} - w + \frac{1-\nu^2}{12}\,\frac{h^2R}{E}\Big(\frac{\partial \varphi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta}\Big) = -\frac{(1-\nu^2)R^2}{Eh}\,p(\alpha,\beta). \tag{2.4}$$

The stress state satisfying the criterion $|\partial^2 f/\partial\alpha^2| \gg |\partial^2 f/\partial\beta^2|$ can be described by a differential equation obtained by applying this strong inequality to the last equation in (1.6):

$$\frac{\partial^4 w}{\partial \alpha^4} + \frac{1-\nu^2}{c^2}\,\Big(1 - \Omega\,\frac{\partial^2}{\partial \alpha^2}\Big)w = \frac{R^4}{D}\Big(1 - \Omega\,\frac{\partial^2}{\partial \alpha^2}\Big)p(\alpha,\beta). \tag{2.5}$$

The expressions for displacements (1.6) and force factors (1.2) also become simplified.

The criterion $|\partial^2 f/\partial\alpha^2| \approx |\partial^2 f/\partial\alpha^2| \gg f$ corresponding to the case of approximately identical variabilities of the stress–strain state in the directions α and β is described by the following system of equations:

$$\nabla^2\nabla^2\nabla^2\nabla^2 w + \frac{1-\nu^2}{c^2}\,(1-\Omega\nabla^2)\,\frac{\partial^4 w}{\partial \alpha^4} = \frac{R^4}{D}\,(1-\Omega\nabla^2)\nabla^2\nabla^2 p(\alpha,\beta),$$

$$\nabla^2\nabla^2 u = -\nu\,\frac{\partial^3 w}{\partial \alpha^3} + \frac{\partial^3 w}{\partial \alpha\,\partial \beta^2}, \qquad \nabla^2\nabla^2 v = -(2+\nu)\,\frac{\partial^3 w}{\partial \alpha^2\partial\,\beta} - \frac{\partial^3 w}{\partial \beta^3}. \tag{2.6}$$

If the variability of the stress state of the shell is so high that we can neglect all terms in the resolving equation (1.6) as compared to terms containing higher derivatives, we obtain the resolving equation

$$\nabla^2\nabla^2 w = (R^4/D)(1-\Omega\nabla^2)p(\alpha,\beta). \tag{2.7}$$

Thus, Eqs. (2.1)–(2.5), (2.6), and (2.7) obtained here generalize the equations of the semi-momentless theory and theory of the edge effect and flexural state to the case of transversely isotropic shells.

3. As an example of applications of some equations obtained here, we consider the action of a radial load, which is piecewise-constant in the longitudinal direction and cosine along the contour, on an infinitely long shell:

$$p(\alpha,\beta) = p_0\theta(\alpha)\cos n\beta \qquad (n = 0, 2, 3, 4, \dots). \tag{3.1}$$

Here p_0 is the amplitude value of pressure; $\theta(\alpha) = 1$ for $|\alpha| \leqslant \alpha_0$ and $\theta(\alpha) = 0$ for $|\alpha| > \alpha_0$.

We present the function $\theta(\alpha)$ in the form of a Fourier integral and obtain

$$p(\alpha,\beta) = \frac{2}{\pi}\,p_0\cos n\beta\int_0^\infty \frac{1}{\lambda}\,\sin\alpha_0\lambda\cos\alpha\lambda\,d\lambda. \tag{3.2}$$

We seek for the solution of Eqs. (1.6) of the general theory of transversely isotropic shells in the form

$$u(\alpha,\beta) = \cos n\beta\int_0^\infty U_n(\lambda)\sin\alpha\lambda\,d\lambda,$$

$$v(\alpha,\beta) = \sin n\beta\int_0^\infty V_n(\lambda)\cos\alpha\lambda\,d\lambda, \qquad w(\alpha,\beta) = \cos n\beta\int_0^\infty W_n(\lambda)\cos\alpha\lambda\,d\lambda.$$

We find the unknown coefficients $U_n(\lambda)$, $V_n(\lambda)$, and $W_n(\lambda)$ by substituting these expansions into (1.6); after that, we can write the expressions in the form of improper integrals for all forces and moments of interest. For brevity, we confine ourselves to the radial displacements:

$$w(\alpha,\beta) = \frac{2}{\pi}\,\frac{p_0 R^4}{D}\,\cos n\beta \int\limits_0^\infty \frac{\tilde{w}_n(\lambda)}{\lambda \tilde{L}_n(\lambda)}\,\sin\alpha_0\lambda\cos\alpha\lambda\,d\lambda; \tag{3.3}$$

$$\tilde{w}_n(\lambda) = [1+\Omega(\lambda^2+n^2)](\lambda^2+n^2)^2,$$
$$\tilde{L}_n(\lambda) = (\lambda^2+n^2)^2(\lambda^2+n^2-1)^2 + 2(1-\nu)\lambda^2(\lambda^2-n^4) + \frac{1-\nu^2}{c^2}\,[1+\Omega(\lambda^2+n^2)]\lambda^4. \tag{3.4}$$

The solution constructed on the basis of equations of the general theory allows us to find the stress–strain state of the transversely isotropic shell with sufficient accuracy. If we consider local loads, however, obtaining of such solutions requires extensive labor-consuming computations, and it seems that deriving formulas convenient for *a priori* estimates is out of the question.

Therefore, we consider application of the methods of asymptotic synthesis of the stress state, where the systems obtained here can play the role of the principal state (2.1)–(2.3), edge effect (2.5), stress state with a high variability (2.6), and flexural state (2.7).

In passing from equations of the general theory (1.6) to equations of the modified semi-momentless theory (2.1)–(2.3) and the theory of the edge effect (2.5), the previously obtained criterion [6] based on ensuring a minimum of the asymptotic error should be considered to be valid.

Concerning the passage from equations of the general theory (1.6) to (2.6), we have to assume in the last relation of (3.4) that

$$\bar{L}_n(\lambda) \approx (\lambda^2+n^2)^4 + (1-\nu^2)[(1+\Omega(\lambda^2+n^2)]\lambda^4/c^2$$

and then find the value of n with which we can neglect the second term, as compared to the first one, i.e.,

$$(\lambda^2+n^2)^4 \gg (1-\nu^2)[1+\Omega(\lambda^2+n^2)]/c^2. \tag{3.5}$$

If we accept, as it was done in [8] in accordance with [3], that $\sqrt{R/h} \gg 1$ rather than $R/h \gg 1$ as in the general theory of shells, we obtain the following equality instead of the strong inequality (3.5):

$$\lambda^8 + 4\lambda^6 n^2 + 6\lambda^4 n^4 + 4\lambda^2 n^6 + n^8 \cong 12(1-\nu^2)(R/h)^2\sqrt{R/h}\,[1+\Omega(\lambda^2+n^2)]\lambda^4. \tag{3.6}$$

To find the criterial value n^*, we have to equate the coefficients at identical powers of λ, as was done in [6]. Then, we obtain

$$n = \sqrt{A\Omega + \sqrt{A^2\Omega^2 + 24A}}\,\Big/(2\sqrt{3}), \qquad A = 12(1-\nu^2)(R/h)^{5/2}. \tag{3.7}$$

The value of n found by Eq. (3.7) and rounded to the nearest integer yields the sought value of the harmonic number $n = n^*$ generalizing an analogous value in the MAS theory of isotropic shells.

4. To illustrate the efficiency of using the above-described approach, we consider the action of a self-balanced system of local radial loads onto an infinitely long shell. Then, instead of (3.2), we have

$$p(\alpha,\beta) = \frac{2}{\pi}\,p_0 \sum_{n=0}^{\infty} \theta_n \cos kn\beta \int\limits_0^\infty \frac{1}{\lambda}\,\sin\alpha_0\lambda\cos\alpha\lambda\,d\lambda, \tag{4.1}$$

where $\theta_n = k\beta_0/\pi$ $(n=0)$ and $\theta_n = 2\sin kn\beta_0/(\pi n)$ $(n = 1,2,3,\ldots,\infty)$; k is the number of loaded domains uniformly distributed over the shell contour at $\alpha = 0$.

The solution constructed for the nth number of the series in the form (3.3), (3.4) on the basis of equations of the general theory can be easily generalized to the case of the local action considered here: it suffices to substitute the value of the coefficient θ_n from (4.1) and apply the operation of summation in terms of n.

Here, we consider only the radial displacement. We obtain its value on the basis of equations of the modified semi-momentless theory (2.1). It almost automatically follows from (3.4) if we take into account Eq. (4.1):

$$w(\alpha,\beta) = \frac{2}{\pi}\,p_0\,\frac{k^4R^2}{Eh} \sum_{n=1}^{n^*} n^4\theta_n \cos kn\beta \int\limits_0^\infty \frac{(1+\Omega k^2n^2)\sin\alpha_0\lambda}{\lambda[(1+\Omega k^2n^2)\lambda^4 + 4\bar{\mu}_n^4]}\,\cos\alpha\lambda\,d\lambda. \tag{4.2}$$

Here $4\bar{\mu}_n^4 = (c^2/(1-\nu^2))k^4n^4(k^2n^2-1)^2$.

We divide the numerator and denominator by $(1+\Omega k^2 n^2)$ and, taking into account that $p_0 = P/(4\alpha_0\beta_0 R^2)$, where P is the total load onto one domain, we obtain

$$\frac{ER}{P}\, w(\alpha,\beta) = \frac{k^4}{2\pi\alpha_0\beta_0}\,\frac{R}{h}\sum_{n=1}^{n^*} n^4\theta_n \cos kn\beta \int\limits_0^\infty \frac{\sin\alpha_0\lambda\cos\alpha\lambda\, d\lambda}{\lambda(\lambda^4+4\mu_n^4)}, \tag{4.3}$$

where $4\mu_n^4 = (c^2/(1-\nu^2))k^2n^2(k^2n^2-1)^2/(1+\Omega k^2n^2)$.

In contrast to the radial displacement (3.4) found from the general theory of transversely isotropic shells, displacement (4.3) found on the basis of modified equations of the semi-momentless theory can be written (after integration) in the form of a simple analytical expression

$$\frac{ER}{P}\, w(\alpha,\beta) = \frac{k^4}{16\pi\alpha_0\beta_0}\,\frac{R}{h}\sum_{n=1}^{n^*}\frac{n^3}{\mu_n^4}\,\sin kn\beta_0 f_n(\alpha)\cos kn\beta, \tag{4.4}$$

where $f_n(\alpha) = 2-\zeta(\alpha_0-\alpha)-\zeta(\alpha_0+\alpha)$, $\alpha \leqslant \alpha_0$, $f_n(\alpha) = \zeta(\alpha-\alpha_0)-\zeta(\alpha+\alpha_0)$, $\alpha > \alpha_0$, and $\zeta(\alpha) = \mathrm{e}^{-\mu_n\alpha}\cos\mu_n\alpha$.

As an example for calculations, we used an infinitely long shell of radius $R = 27.25$ mm and thickness $h = 0.82$ mm. The loaded region was characterized by the parameters $\alpha_0 = \beta_0 = 0.005$, and the degree of anisotropy was described by the ratio $E/G^\gamma = 80$. The shell was affected by two oppositely directed diametral forces; hence, $k = 2$.

We present the results of calculation by Eq. (4.3) for the radial displacement under a forced action: $(ER/P)w(0,0) = 4659$ for $\Omega = 0$, i.e., with no allowance for the transverse displacement, and $(ER/P)w(0,0) = 5275$ with allowance for the transverse displacement.

In our case, the allowance for the transverse displacement leads to an increase in the radial displacement approximately by 12%. Note, the solution proposed is less labor-consuming than that in the general theory of shells. The local edge effect here corrects the displacement only slightly, and the role of the flexural state is almost invisible.

It seems of interest to compare the displacement found by Eq. (4.3) for $\Omega = 0$ [$w(0,0) = 0.0854$ mm] with the solution found with the use of the general theory for $\Omega = 0$ [$w(0,0) = 0.0879$ mm] and with the precision experiment [8] for a shell with the above-cited dimensions [$w(0,0) = 0.0900$ mm].

Note, the value of the displacement found by formula (4.3) for $E/G^\gamma = 80$ for the same shell is $w(0,0) = 0.0967$ mm. This indicates good correlation of all numerical results and reliability of the equations proposed and the methods of solving them.

REFERENCES

1. V. Z. Vlasov, "General theory of shells and its applications in engineering," in: *Selected Papers* [in Russian], Vol. 1, Izd. Akad. Nauk SSSR, Moscow (1962).
2. S. A. Ambartsumyan, *General Theory of Anisotropic Shells* [in Russian], Nauka, Moscow (1974).
3. V. V. Novozhilov, *Theory of Thin Shells* [in Russian], Sudpromgiz, Leningrad (1962).
4. A. L. Goldenweiser, *Theory of Elastic Thin Shells* [in Russian], Nauka, Moscow (1976).
5. I. F. Obraztsov, B. V. Nerubailo, and I. V. Andrianov, *Asymptotic Methods in Mechanical Engineering of Thin-Walled Structures* [in Russian], Mashinostroenie, Moscow (1991).
6. I. F. Obraztsov and B. V. Nerubailo, "Methods of stress-state synthesis in the theory of shells," *Dokl. Akad. Nauk SSSR*, No. 3, 54–56 (1983).
7. V. Z. Vlasov, "Constitutive differential equations of the general theory of elastic shells," *Prikl. Mat. Mekh.*, **8**, No. 2, 109–140 (1944).
8. B. V. Nerubailo, *Local Problems of Strength of Cylindrical Shells* [in Russian], Mashinostroenie, Moscow (1983).